

# THE ACTION OF HECKE OPERATORS ON HYPERGEOMETRIC FUNCTIONS

VICTOR H. MOLL, SINAI ROBINS, AND KIRK SOODHALTER

**ABSTRACT.** We study the action of the Hecke operators  $U_n$  on the set of hypergeometric functions, as well as on formal power series. We show that the spectrum of these operators on the set of hypergeometric functions is the set  $\{n^a : n \in \mathbb{N} \text{ and } a \in \mathbb{Z}\}$ , and that the polylogarithms play a dominant role in the study of the eigenfunctions of the Hecke operators  $U_n$  on the set of hypergeometric functions. As a corollary of our results on simultaneous eigenfunctions, we also obtain an apriori unrelated result regarding the behavior of completely multiplicative hypergeometric coefficients.

## 1. INTRODUCTION

For each  $n \in \mathbb{N}$ , the space of formal power series

$$(1.1) \quad \mathfrak{F} := \left\{ f(x) = \sum_{k=0}^{\infty} c_k x^k : c_k \in \mathbb{C} \right\}$$

admits the action of the linear operators

$$(1.2) \quad (U_n f)(x) := \sum_{k=0}^{\infty} c_{nk} x^k$$

and

$$(1.3) \quad (V_n f)(x) := f(x^n) = \sum_{k=0}^{\infty} c_k x^{nk}.$$

The spectral properties of these operators become more interesting when one considers their action on spaces with additional structure.

Historically, Hecke studied the vector spaces of modular forms of a fixed weight (see [2]), in which the set  $\mathfrak{F}$  above is replaced by the space  $\mathfrak{M}_k$  defined by analytic functions in the upper half-plane  $\mathbb{H} := \{\tau : \text{Im } \tau > 0\}$ , that satisfy the condition

$$(1.4) \quad f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau),$$

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for every matrix in the modular group

$$(1.5) \quad \Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ with } ad - bc = 1 \right\}.$$

These forms are also required to have an expansion at  $\tau = i\infty$ , or equivalently a Taylor series about  $q = 0$ :

$$(1.6) \quad f(\tau) = \sum_{n=0}^{\infty} c(n)q^n,$$

expressed in terms of the parameter  $q = \exp(2\pi i\tau)$ .

Hecke introduced a family of operators  $T_n$ , for  $n \in \mathbb{N}$ , which map the space  $\mathfrak{M}_k$  into itself. The standard definition is

$$(1.7) \quad (T_n f)(\tau) := n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{n\tau + bd}{d^2}\right),$$

that in the special case  $n = p$  prime, becomes

$$(1.8) \quad (T_p f)(\tau) = p^{k-1} f(p\tau) + \frac{1}{p} \sum_{b=0}^{p-1} f\left(\frac{\tau + b}{p}\right).$$

In terms of the Fourier expansion of  $f \in \mathfrak{M}_k$ , given by

$$(1.9) \quad f(\tau) = \sum_{m=0}^{\infty} c(m)q^m,$$

the action of  $T_n$  is

$$(1.10) \quad (T_n f)(\tau) = \sum_{m=0}^{\infty} \gamma_n(m)q^m,$$

where

$$(1.11) \quad \gamma_n(m) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right).$$

In particular, when  $n = p$  is prime, we have

$$(1.12) \quad \gamma_p(m) = \begin{cases} c(pm) + p^{k-1} c\left(\frac{m}{p}\right) & \text{if } p|m, \\ c(pm) & \text{if } p \nmid m. \end{cases}$$

History has shown that the study of Hecke operators is one of the most important tools in modern Number Theory, yielding results about the uniform distributions of points, the eigenvalues of Laplacians on various domains, the asymptotic analysis of Fourier coefficients of modular forms, and other branches of Number Theory.

Interesting results were obtained in the last decade when the space of modular forms was replaced with the space of rational functions (see [9], [5], and [11]). For example, the spectral properties of the operator  $U_n$  acting on rational functions were completely characterized, and corollaries about completely multiplicative functions that satisfy linear recurrence sequences were obtained (see [9]).

For the space  $\mathfrak{R}$  of rational functions, the coefficients  $a_n$  in (1.1) are the Taylor coefficients of  $A/B \in \mathfrak{R}$ , with  $B(x) = 1 + \alpha_1 x + \cdots + \alpha_d x^d$  and  $A$  a polynomial in

$x$ , of degree less than  $d$ . These coefficients  $a_n$  are known to satisfy the recurrence relation

$$(1.13) \quad a_{n+d} = -\alpha_1 a_{n+d-1} - \cdots - \alpha_d a_n,$$

see [12] for details. Thus the study of these coefficients employs the theory of linear recurrence sequences and their explicit solutions. One of the main results in [9] is the complete determination of the spectrum of  $U_n$  acting on  $\mathfrak{R}$ :

$$(1.14) \quad \text{spec}(U_n) = \{\pm n^k : k \in \mathbb{N}\} \cup \{0\}.$$

Recent work has produced a description of the corresponding rational eigenfunctions (see [11, 4]).

In this paper, we consider the action of  $U_n$  on the *set* of hypergeometric functions

$$(1.15) \quad \mathfrak{H} := \left\{ \sum_{k=0}^{\infty} c_k x^k : \frac{c_{k+1}}{c_k} \text{ is a rational function of } k \right\}.$$

We emphasize here that  $\mathfrak{H}$  is a set rather than a vector space, because the sum of two hypergeometric functions is *not* in general another hypergeometric function. Nevertheless, this set includes most of the classical functions as well as all functions of the form

$$(1.16) \quad \sum_{k=0}^{\infty} R(k) x^k,$$

where  $R$  is a rational function.

Every hypergeometric function that we consider has a canonical Taylor series representation of the form

$$(1.17) \quad {}_pF_q(\mathbf{a}, \mathbf{b}; x) := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

where  $\mathbf{a} := (a_1, a_2, \dots, a_p) \in \mathbb{C}^p$  and  $\mathbf{b} := (b_1, b_2, \dots, b_q) \in \mathbb{C}^q$  are the *parameters* of  ${}_pF_q$ . These parameters satisfy  $-b_i \notin \mathbb{N}$ . Here we use the standard notation for the ascending factorial symbol  $(c)_k := c(c+1)(c+2) \cdots (c+k-1)$ , and  $(c)_0 := 1$ . For example  $(1)_k = k!$ , and  $(0)_k = 0$ .

Hypergeometric functions include

$$(1.18) \quad f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k^2 + 1},$$

as well as most of the elementary functions. For example, the hypergeometric representation of the exponential function is given by  $e^x = {}_1F_1(a, a; x)$  for any nonzero  $a \in \mathbb{C}$ . Similarly, the error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

can also be represented as a hypergeometric function, namely

$$\text{erf}(x) = \frac{2x}{\sqrt{\pi}} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right).$$

For a more complete discussion of hypergeometric functions, see [1].

In Section 5 we describe the action of the Hecke operator  $U_n$  on hypergeometric functions. To state the results, define

$$\mathfrak{F}_{(p,q)}^j := \{x^j {}_pF_q(\mathbf{a}, \mathbf{b}; x) : \mathbf{a} \in \mathbb{C}^p, \mathbf{b} \in \mathbb{C}^q\},$$

for fixed  $j \in \mathbb{N}$ , and fixed  $p, q \in \mathbb{N}$ . This is the set of all hypergeometric functions that vanish to order  $j$  at the origin, and have hypergeometric coefficients

$$\frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k}$$

with  $p$  ascending factorials in the numerator and  $q$  ascending factorials in the denominator. Observe that

$$\begin{aligned} \mathfrak{H} &:= \{x^j {}_pF_q(\mathbf{a}, \mathbf{b}; x) : \text{for some } j, p, q \in \mathbb{N}, \text{ and some parameters } \mathbf{a} \in \mathbb{C}^p, \mathbf{b} \in \mathbb{C}^q\} \\ &= \bigcup_{j,p,q \in \mathbb{N}} \mathfrak{F}_{(p,q)}^j. \end{aligned}$$

We establish first the identities

$$U_n(x^j {}_pF_q(\mathbf{a}, \mathbf{b}; x)) = x^{j/n} \sum_{k=0}^{\infty} n^{nk(p-q-1)} \frac{(c_1)_k (c_2)_k \cdots (c_{np})_k}{(d_1)_k (d_2)_k \cdots (d_{n(q+1)-1})_k} \frac{x^k}{k!} \in \mathfrak{F}_{(p_1, q_1)}^j,$$

when  $n$  divides  $j$  and

$$U_n(x^j {}_pF_q(\mathbf{a}, \mathbf{b}; x)) = x^{1+\lfloor j/n \rfloor} \sum_{k=0}^{\infty} n^{nk(p-q-1)} \frac{(c_1)_k \cdots (c_{pn})_k}{(d_1)_k \cdots (d_{(q+1)n-1})_k} \frac{x^k}{k!} \in \mathfrak{F}_{(p_1, q_1)}^j.$$

if  $n$  does not divide  $j$ . Here  $p_1 = np$ ,  $q_1 = n(q+1) - 1$  and the new parameters  $\mathbf{c}, \mathbf{d}$  are given in (5.3) and (5.4) respectively. In particular, we observe in Section 5 that  $U_n$  maps  $\mathfrak{F}_{(p,q)}^j$  into itself if and only if  $p = q + 1$ . These are the *balanced* hypergeometric functions. Therefore, an eigenfunction of  $U_n$  must have  $p = q + 1$ .

The eigenfunctions of  $U_n$  on the space of formal power series are described in Section 4. We consider solutions of  $U_n f = \lambda f$ , for  $f(x) = x^j \sum_{k=0}^{\infty} a_k x^k$  and show that if  $n$  divides  $j$ , then it follows that  $j = 0$ ,  $\lambda = 1$  and the eigenfunction  $f$  must reduce to the rational function  $\frac{1}{1-x}$ . On the other hand, in the case that  $n$  does not divide  $j$ , we show that  $j$  must be 1 and the eigenvalue  $\lambda$  must be of the form  $n^a$ , with  $a \in \mathbb{Z}$ .

One of the main results here is the complete characterization of the spectrum of  $U_n$  on hypergeometric functions, yielding the result that

$$(1.19) \quad \text{spec}(U_n) = \{n^k : k \in \mathbb{Z}\} \cup \{0\}.$$

As a corollary, we obtain a number-theoretic characterization of all completely multiplicative functions that are also hypergeometric ratios of ascending factorials.

An ultimate goal is to determine the spectrum of all linear combinations of hypergeometric series, but this seems far from the reach of current technology. Thus we focus first on the action of the Hecke operators on a single hypergeometric series.

2. A NATURAL INNER PRODUCT ON  $\mathfrak{H}$ 

The set of all hypergeometric functions  $\mathfrak{H}$  can be endowed with a natural inner product defined by

$$(2.1) \quad \langle f, g \rangle_R := \oint_{|z|=R} f(w) \overline{g(w)} \frac{dw}{w}.$$

We fix a real number  $R$  with  $0 < R < 1$ , so that this inner product is now a function of  $R$ , and as we shall see shortly it is in fact a real analytic function of  $R$ . Moreover, we shall also see below that  $V_n$  is the natural conjugate linear operator to  $U_n$  with respect to this inner product. This fact is our motivation for introducing the linear operator  $V_n$ .

The next result describes this inner product in terms of the Taylor series expansions of  $f$  and  $g$ .

**Lemma 2.1.** If  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  and  $g(z) = \sum_{n=0}^{\infty} d_n z^n$ , then

$$(2.2) \quad \langle f, g \rangle_R = 2\pi i \sum_{n=0}^{\infty} c_n \overline{d_n} R^{2n}.$$

*Proof.* By definition we have

$$\begin{aligned} \langle f, g \rangle_R &= \oint_{|w|=R} \sum_{n=0}^{\infty} c_n w^n \times \sum_{m=0}^{\infty} \overline{d_m} \overline{w}^m \frac{dw}{w} \\ &= \oint_{|w|=R} \sum_{n,m=0}^{\infty} c_n \overline{d_m} w^{n-m-1} R^{2m} dw, \end{aligned}$$

because  $\overline{w} = R^2/w$  on the circle of integration. Cauchy's integral formula shows that  $n = m$ , for otherwise the ensuing line integral is zero. This condition gives us the desired result.  $\square$

This inner product makes sense even for formal power series, although we restrict our attention to the set  $\mathfrak{H}$  of hypergeometric functions. On  $\mathfrak{H}$ , the inner product of any two hypergeometric functions is in fact a hypergeometric function of  $R$ , as is easily seen by noting that the product of two hypergeometric coefficients is another hypergeometric coefficient.

To further develop the algebra of the operators  $U_n$  and  $V_n$ , we now show that  $V_n$  is the natural conjugate linear operator for  $U_n$ , relative to the inner product introduced above.

**Lemma 2.2.** Let  $f, g \in \mathfrak{H}$ . Then

$$(2.3) \quad \langle U_n f, g \rangle_R = \langle f, V_n g \rangle_R.$$

*Proof.* Start with

$$\begin{aligned} \langle U_n f, g \rangle_R &= \left\langle \sum_{k=0}^{\infty} c_{kn} z^k, \sum_{k=0}^{\infty} d_k z^k \right\rangle_R \\ &= \sum_{k=0}^{\infty} c_{kn} \overline{d_k} R^{2k}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\langle f, V_n g \rangle_R &= \left\langle \sum_{k=0}^{\infty} c_k z^k, \sum_{k=0}^{\infty} d_k z^{kn} \right\rangle_R \\ &= \sum_{k=0}^{\infty} c_k \overline{h_k} R^{2k},\end{aligned}$$

where we define

$$h_k = \begin{cases} 0 & \text{if } n \text{ does not divide } k \\ d_{k/n} & \text{if } n \text{ divides } k. \end{cases}$$

It follows that

$$(2.4) \quad \langle f, V_n g \rangle_R = \sum_{k=0}^{\infty} c_{kn} \overline{d_k} (R^n)^{2k} = \langle U_n f, g \rangle_{R^n}.$$

□

Recall that Hadamard introduced an inner product in the space of formal power series  $\mathfrak{F}$ , by

$$(2.5) \quad (f * g)(x) := \sum_{k=0}^{\infty} c_k d_k x^k$$

This product can be retrieved as a special case of  $\langle, \rangle_R$ ; namely,

$$(2.6) \quad \left\langle \sum_{k=0}^{\infty} c_k x^k, \overline{\sum_{k=0}^{\infty} d_k x^k} \right\rangle_{R^{1/2}} = \sum_{k=0}^{\infty} c_k d_k R^k.$$

Thus the Hadamard product is completely equivalent to our inner product.

### 3. ELEMENTARY PROPERTIES OF THE OPERATORS $U_n$ AND $V_n$

In this section we describe elementary properties of the operators defined in (1.2) and (1.3).

**Theorem 3.1.** Let  $m, n \in \mathbb{N}$ . Then

- a)  $U_n \circ U_m = U_m \circ U_n = U_{nm}$ ,
- b)  $V_n \circ V_m = V_m \circ V_n = V_{nm}$ ,
- c)  $U_n \circ V_n = \text{Id}$ ,
- d)  $U_n \circ V_m = V_{m/\gcd(m,n)} \circ U_{n/\gcd(m,n)}$ . In particular, if  $m$  and  $n$  are relatively prime, then  $U_n$  and  $V_m$  commute.

*Proof.* Let  $f \in \mathfrak{F}$  be a formal power series with coefficients  $c_k$ . The first two properties follow directly from

$$U_n U_m f(x) = U_n \sum_{k=0}^{\infty} c_{mk} x^k = \sum_{k=0}^{\infty} c_{nmk} x^k = U_{nm} f(x),$$

and similarly for  $V_n V_m$ . To establish the third property observe that

$$\begin{aligned} U_n V_n f(x) &= U_n V_n \left( \sum_{k=0}^{\infty} c_k x^k \right) = U_n \left( \sum_{k=0}^{\infty} c_k x^{kn} \right) \\ &= \sum_{k=0}^{\infty} c_k x^k = f(x). \end{aligned}$$

Finally,

$$U_n \circ V_m \sum_{k=0}^{\infty} c_k x^k = U_n \sum_{k=0}^{\infty} c_k x^{mk}.$$

To simplify this, let

$$(3.1) \quad d_k = \begin{cases} c_{k/m} & \text{if } m \text{ divides } k \\ 0 & \text{if } m \text{ does not divide } k, \end{cases}$$

to write

$$U_n \circ V_m \left( \sum_{k=0}^{\infty} c_k x^k \right) = U_n \left( \sum_{k=0}^{\infty} d_k x^k \right) = \sum_{k=0}^{\infty} d_{nk} x^k.$$

Now observe that  $m$  divides  $kn$  if and only if  $m/\gcd(m, n)$  divides  $k$ . Therefore, we define

$$N = \frac{n}{\gcd(m, n)}, \quad M = \frac{m}{\gcd(m, n)},$$

and we have the following sum on  $k$  (where the sums below are defined for  $k$  going from 0 to  $\infty$ , subject to the constraints given):

$$(3.2) \quad \sum_k d_{nk} x^k = \sum_{m|nk} d_{nk} x^k = \sum_{m|nk} c_{\frac{nk}{m}} x^k = \sum_{M|Nk} c_{\frac{Nk}{M}} x^k = \sum_{M|k} c_{N \frac{k}{M}} x^k.$$

We now let  $k = iM$ , and sum over  $i \geq 0$ , to get:

$$\sum_{M|k} c_{N \frac{k}{M}} x^k = \sum_{i=0}^{\infty} c_{iN} x^{iM}.$$

Now define  $h_i = c_{iN}$  to obtain

$$\sum_{i=0}^{\infty} h_i x^{iM} = V_M \left( \sum_{i=0}^{\infty} h_i x^i \right) = V_M \left( \sum_{i=0}^{\infty} c_{iN} x^i \right) = V_M \circ U_N \left( \sum_{i=0}^{\infty} c_i x^i \right),$$

and we have established part (d).  $\square$

We present now an alternate proof for Theorem 3.1. To do this, we must prove an intermediate result.

**Lemma 3.1.** (*Associativity of  $U$  and  $V$* )

For all  $k, j, m \in \mathbb{N}$ ,  $U_{kj} \circ V_m = U_k \circ (U_j \circ V_m)$

*Proof.* Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ . For this proof, we will evaluate both operators and show they are the same operation. First, for  $U_{kj} \circ V_m$ , we have that

$$(U_{kj} \circ V_m) f(z) = U_{kj} f(z^m) = U_{kj} \left( \sum_{n \geq 0} a_n z^{mn} \right)$$

Now we let

$$b_i = \begin{cases} a_{i/m} & \text{if } m \text{ divides } i \\ 0 & \text{otherwise} \end{cases}$$

so that we can write

$$U_{kj}(\sum_{n \geq 0} a_n z^{mn}) = U_{kj}(\sum_{i \geq 0} b_i z^i) = \sum_{i \geq 0} b_{(kj)i} z^i.$$

Now, for  $U_k \circ (U_j \circ V_m)$ , we have that

$$U_k \circ (U_j \circ V_m)(\sum_{n \geq 0} a_n z^n) = U_k(U_j(\sum_{n \geq 0} a_n z^{mn})) = U_k(\sum_{i \geq 0} c_i z^i)$$

with

$$c_i = \begin{cases} a_n & \text{if } ij = mn \text{ for } n \in \mathbb{N} \cup \{0\} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$U_k(\sum_{i \geq 0} c_i z^i) = \sum_{i \geq 0} c_{ki} z^i$$

We complete the proof by noting that  $b_{(kj)i} = c_{ki}$ .  $\square$

*Proof.* (Theorem 3.1 *alternative*) We can write  $U_n \circ V_m = (U_{n/\gcd(m,n)} \circ U_{\gcd(m,n)}) \circ (V_{\gcd(m,n)} \circ V_{m/\gcd(m,n)})$  by parts (a) and (b) of the theorem.

Now, using associativity, we can write

$$= U_{n/\gcd(m,n)} \circ (U_{\gcd(m,n)} \circ V_{\gcd(m,n)}) \circ V_{m/\gcd(m,n)}$$

By part (c) of the theorem, we have that  $U_{\gcd(m,n)} \circ V_{\gcd(m,n)} = I$ , and we are left with  $U_n \circ V_m = U_{m/\gcd(m,n)} \circ V_{\gcd(m,n)}$ . Thus, part (d) of the theorem is proven.  $\square$

#### 4. THE ACTION OF $U_n$ ON FORMAL POWER SERIES

We now determine an expression for the action of the operator  $U_n$  acting on formal power series where we allow the first few coefficients to vanish. This result will be employed in our study of spectral properties of  $U_n$  acting on hypergeometric functions. For the rest of this section, all functions are assumed to be formal power series.

**Theorem 4.1.** Let  $j, n \in \mathbb{N}$ . Then

$$U_n \left( x^j \sum_{k \geq 0} a_k x^k \right) = \begin{cases} x^{1+\lfloor j/n \rfloor} \sum a_{n(k+1-\lfloor j/n \rfloor)} x^k & \text{if } n \text{ does not divide } j \\ x^{j/n} \sum a_{kn} x^k & \text{if } n \text{ divides } j, \end{cases}$$

where the sums are over  $k \geq 0$ .

*Proof.* First observe that

$$\begin{aligned} U_n \left( x^j \sum_{k \geq 0} a_k x^k \right) &= U_n \left( \sum_{k \geq 0} a_k x^{k+j} \right) \\ &= U_n \left( \sum_{k \geq j} a_{k-j} x^k \right) \end{aligned}$$



and define

$$(4.1) \quad b_k = \begin{cases} 0 & 0 \leq k < j \\ a_{k-j} & k \geq j \end{cases}$$

to write

$$(4.2) \quad U_n \left( x^j \sum_{k \geq 0} a_k x^k \right) = U_n \left( \sum_{k \geq 0} b_k x^k \right) = \sum_{k \geq 0} b_{kn} x^k.$$

The discussion is divided in two cases according to whether  $n$  divides  $j$  or not.

**Case 1:**  $n$  does not divide  $j$ . Then the restriction  $kn \geq j$  in (4.1) is equivalent to  $k \geq \lfloor \frac{j}{n} \rfloor + 1$ . Thus,

$$\begin{aligned} \sum_{k \geq 0} b_{kn} x^k &= \sum_{k=0}^{\lfloor \frac{j}{n} \rfloor} b_{kn} x^k + \sum_{k=\lfloor \frac{j}{n} \rfloor + 1}^{\infty} b_{kn} x^k \\ &= \sum_{k=\lfloor \frac{j}{n} \rfloor + 1}^{\infty} a_{kn-j} x^k. \end{aligned}$$

Now let  $i = k - \lfloor \frac{j}{n} \rfloor - 1$ , to obtain

$$(4.3) \quad \sum_{k \geq 0} b_{kn} x^k = x^{\lfloor \frac{j}{n} \rfloor + 1} \sum_{i \geq 0} a_{ni+n\lfloor \frac{j}{n} \rfloor+n-j} x^i.$$

Now use  $\frac{j}{n} = \lfloor \frac{j}{n} \rfloor + \{ \frac{j}{n} \}$  to obtain

$$(4.4) \quad U_n \left( x^j \sum_{k \geq 0} a_k x^k \right) = x^{\lfloor \frac{j}{n} \rfloor + 1} \sum_{i \geq 0} a_{n(i+1-\{ \frac{j}{n} \})} x^i.$$

This is the result when  $n$  does not divide  $j$ .

**Case 2:**  $n$  divides  $j$ . Then  $kn \geq j$  is now equivalent to  $k \geq \frac{j}{n} = \lfloor \frac{j}{n} \rfloor$  and

$$\begin{aligned} \sum_{k \geq 0} b_{kn} x^k &= \sum_{k \geq \lfloor j/n \rfloor} b_{kn} x^k \\ &= \sum_{k \geq \lfloor j/n \rfloor} a_{kn-j} x^k \\ &= x^{\lfloor j/n \rfloor} \sum_{i \geq 0} a_{ni} x^i \end{aligned}$$

so that

$$(4.5) \quad U_n \left( x^j \sum_{k \geq 0} a_k x^k \right) = x^{\lfloor \frac{j}{n} \rfloor} \sum_{\nu \geq 0} a_{n\nu} x^\nu.$$

This concludes the proof.  $\square$

The previous expressions for  $U_n$  are now used to derive some elementary properties of its eigenfunctions on the space of formal power series.

**Proposition 4.2.** Assume  $U_n$  has an eigenfunction of the form

$$(4.6) \quad f(x) = x^j \sum_{k=0}^{\infty} a_k x^k,$$

with eigenvalue  $\lambda$ . If  $n$  divides  $j$ , then we conclude that  $j = 0$  and  $\lambda = 1$ . If  $n$  does not divide  $j$ , then we conclude that  $j = 1$ .

*Proof.* Assume  $n$  divides  $j$  and match the leading order terms of  $f$  and  $U_n f$ . Theorem 4.1 shows that  $x^j = x^{j/n}$  yielding  $j = 0$ . Now compare the constant term in the eigenvalue equation to get  $\lambda = 1$ . In the case  $n$  does not divide  $j$  the same comparison yields

$$(4.7) \quad j = 1 + \lfloor j/n \rfloor.$$

This implies  $j = 1$ . Indeed, let  $j = \alpha n + \beta$  with  $0 < \beta < n$ . Then we have  $j = 1 + \alpha$ , and this yields

$$(4.8) \quad 1 - \beta = \alpha(n - 1).$$

It follows that  $\beta = 1$  and  $\alpha = 0$ , otherwise both sides of (4.8) have different signs. We conclude that  $j = 1 + \alpha = 1$ .  $\square$

Hence even for a formal power series  $f$ , we see that the assumption that  $f$  is an eigenfunction of the Hecke operator  $U_n$  imposes the restriction that  $f$  can only vanish to order zero or one.

For the sake of completeness, we describe the trivial eigenfunctions of the composition of operators  $U_n \circ V_n$  and  $V_n \circ U_n$ . Theorem 3.1 shows that  $U_n \circ V_n$  is the identity. The next result describes the composition  $V_n \circ U_n$ .

**Theorem 4.3.** The only eigenvalue of  $V_n \circ U_n$  is  $\lambda = 1$ . Moreover, given any formal power series  $f(x) = \sum_{k=0}^{\infty} b_k x^k$ , the function  $g(x) = \sum_{k=0}^{\infty} a_k x^k$ , with

$$(4.9) \quad a_k = \begin{cases} b_k & \text{if } n \text{ divides } k \\ 0 & \text{if } n \text{ does not divide } k \end{cases}$$

is an eigenfunction of  $V_n \circ U_n$ , with eigenvalue 1.

*Proof.* The result follows directly from the identity

$$(4.10) \quad (V_n \circ U_n) \left( \sum_{k=0}^{\infty} a_k x^k \right) = \sum_{k=0}^{\infty} a_{kn} x^k.$$

$\square$

## 5. THE HYPERGEOMETRIC FUNCTIONS

In this section we use Theorem 4.1 to describe the action of  $U_n$  on the set  $\mathfrak{H}$  of all hypergeometric functions. We recall that a hypergeometric function is defined by

$$(5.1) \quad {}_pF_q(\mathbf{a}, \mathbf{b}; x) := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

where  $\mathbf{a} := (a_1, a_2, \dots, a_p)$  and  $\mathbf{b} := (b_1, b_2, \dots, b_q)$  are the parameters of  ${}_pF_q$ . These parameters are non-zero complex numbers. We begin by stating explicitly the action of  $U_n$  on  $\mathfrak{F}_{(p,q)}^j$  as the main result of this section.

**Theorem 5.1.** Let  $j, n \in \mathbb{N}$ . The action of  $U_n$  on the class  $\mathfrak{F}_{(p,q)}^j$ , that is, on functions of the form

$$(5.2) \quad f_{p,q,j} = x^j {}_pF_q(\mathbf{a}, \mathbf{b}; x)$$

is characterized as follows.

If  $n$  divides  $j$ , we have

$$U_n \left( x^j \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k \right) = x^{j/n} \sum_{k=0}^{\infty} \frac{(c_1)_k (c_2)_k \cdots (c_{np})_k}{(d_1)_k (d_2)_k \cdots (d_{n(q+1)-1})_k} \frac{x_1^k}{k!},$$

where we define the parameters

$$(5.3) \quad \begin{aligned} c_{in+l} &= \frac{a_{i+1} + l - 1}{n}, & \text{for } 0 \leq i \leq p-1, 1 \leq l \leq n \\ d_{in+l} &= \frac{b_{i+1} + l - 1}{n}, & \text{for } 0 \leq i \leq q, 1 \leq l \leq n, \end{aligned}$$

and  $j_1 = j/n$ . The new variable is  $x_1 = n^{n(p-q-1)}x$ .

If  $n$  does not divide  $j$ , we have

$$U_n \left( x^j \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k \right) = x^{1+\lfloor j/n \rfloor} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_{pn})_k}{(d_1)_k \cdots (d_{(q+1)n-1})_k} \frac{x_1^k}{k!},$$

where we now define the parameters

$$(5.4) \quad \begin{aligned} c_{in+l} &= \frac{a_{i+1} + r + l}{n}, & \text{for } 0 \leq i \leq p-1, 1 \leq l \leq n, \\ d_{in+l} &= \frac{b_{i+1} + r + l}{n}, & \text{for } 0 \leq i \leq q, 1 \leq l \leq n, \end{aligned}$$

with  $j_2 = 1 + \lfloor j/n \rfloor$  and  $r = n(1 - \{j/n\}) - 1$ . The new variable  $x_1$  is defined as above.

Before proving this theorem, we first need to state some intermediate results. The next lemma allows for a simplification of the ascending factorial function on an arithmetic progression of indices.

**Lemma 5.2.** Let  $k, n \in \mathbb{N}$  and  $a \in \mathbb{R}$ . Then

$$(a)_{kn} = n^{kn} \prod_{j=0}^{n-1} \left( \frac{a+j}{n} \right)_k.$$

*Proof.* Start with

$$(a)_{kn} = \prod_{i=0}^{kn-1} (a+i) = n^{kn} \prod_{i=0}^{kn-1} \left( \frac{a}{n} + \frac{i}{n} \right),$$

and then collect terms according to classes modulo  $n$ . □

In order to evaluate

$$U_n(x^j {}_pF_q(\mathbf{a}, \mathbf{b})) = U_n\left(x^j \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k (b_{q+1})_k} x^k\right),$$

where we have used  $k! = (1)_k$  and defined  $b_{q+1} = 1$ , we observe that by Theorem 4.1 the discussion should be divided into two cases according to whether or not  $n$  divides  $j$ .

**Case 1:**  $n$  divides  $j$ . Theorem 4.1 yields

$$(5.5) \quad U_n(x^j {}_pF_q(\mathbf{a}, \mathbf{b})) = x^{j/n} \sum_{k=0}^{\infty} \frac{(a_1)_{kn} \cdots (a_p)_{kn}}{(b_1)_{kn} \cdots (b_{q+1})_{kn}} x^k,$$

and using Lemma 5.2 we can write this as

$$U_n(x^j {}_pF_q(\mathbf{a}, \mathbf{b})) = x^{j/n} \sum_{k=0}^{\infty} n^{(p-q-1)kn} \left( \prod_{j=1}^p \prod_{i=0}^{n-1} \left( \frac{a_j + i}{n} \right)_k \right) \times \left( \prod_{j=1}^{q+1} \prod_{i=0}^{n-1} \left( \frac{b_j + i}{n} \right)_k \right)^{-1} x^k.$$

Now recall that  $b_{q+1} = 1$ , so the  $d$ -parameters corresponding to  $i = q$  from the definition (5.4) are  $1/n, 2/n, \dots, (n-1)/n, 1$ . The total number of  $d$ -parameters is now reduced by 1, in order to write the result in the canonical hypergeometric form:

$$U_n(x^j {}_pF_q(\mathbf{a}, \mathbf{b})) = x^{j/n} \sum_{k=0}^{\infty} n^{(p-q-1)kn} \frac{(c_1)_k (c_2)_k \cdots (c_{np})_k}{(d_1)_k (d_2)_k \cdots (d_{n(q+1)-1})_k} \frac{x^k}{k!}.$$

**Lemma 5.3.** The parameters  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  and  $\mathbf{d}$  satisfy

$$\sum_{i=1}^{np} c_i - \sum_{i=1}^{n(q+1)-1} d_i = \sum_{i=1}^p a_i - \sum_{i=1}^q b_i + \frac{(n-1)}{2}(p-q-1).$$

*Proof.* The new parameters are

$$\frac{a_1}{n}, \frac{a_1+1}{n}, \dots, \frac{a_1+n-1}{n}, \frac{a_p}{n}, \dots, \frac{a_p+n-1}{n}$$

and

$$\frac{b_1}{n}, \frac{b_1+1}{n}, \dots, \frac{b_1+n-1}{n}, \frac{b_{q+1}}{n} = \frac{1}{n}, \dots, \frac{b_{q+1}+n-2}{n} = \frac{n-1}{n}$$

and the identity is now easy to check.  $\square$

**Corollary 5.4.** If  $p = q+1$ , then  $\sum a_i - \sum b_i$  is preserved under the action of  $U_n$ .

**Case 2:**  $n$  does not divide  $j$ . Proposition 4.1 now gives

$$(5.6) \quad U_n\left(x^j \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k\right) = x^{1+\lfloor j/n \rfloor} \sum_{k=0}^{\infty} \frac{(a_1)_N \cdots (a_p)_N}{(b_1)_N \cdots (b_{q+1})_N} x^k,$$

where  $b_{q+1} = 1$  and we define  $N = n(k+1 - \{j/n\})$ . Observe that  $0 < \{j/n\} < 1$ , thus  $nk < N < n(k+1)$ . The next result simplifies the Pochhammer symbols.

**Lemma 5.5.** Let  $a \in \mathbb{C}$  and  $j, n \in \mathbb{N}$  with  $j$  not divisible by  $n$ . Define  $N = n(k+1 - \{j/n\})$  and  $r = n(1 - \{j/n\}) - 1$ . Then

$$(5.7) \quad (a)_N = n^{nk} (a)_{r+1} \prod_{i=r+1}^{r+n} \left( \frac{a+i}{n} \right)_k$$

*Proof.* We begin with

$$\begin{aligned} (a)_N &= a(a+1)(a+2) \cdots (a+N-1) \\ &= n^N \left[ \frac{a}{n} \left( \frac{a}{n} + \frac{1}{n} \right) \cdots \left( \frac{a}{n} + \frac{N-1}{n} \right) \right]. \end{aligned}$$

We now regroup terms modulo  $n$  as follows:

$$\begin{aligned} n^{-N} (a)_N &= \left( \frac{a}{n} \right) \cdot \left( \frac{a}{n} + 1 \right) \cdots \left( \frac{a}{n} + k - 1 \right) \\ &\times \left( \frac{a}{n} + \frac{1}{n} \right) \cdot \left( \frac{a}{n} + \frac{1}{n} + 1 \right) \cdots \left( \frac{a}{n} + \frac{1}{n} + k - 1 \right) \\ &\cdots \\ &\times \left( \frac{a}{n} + \frac{n-1}{n} \right) \cdot \left( \frac{a}{n} + \frac{n-1}{n} + 1 \right) \cdots \left( \frac{a}{n} + \frac{n-1}{n} + k - 1 \right) \\ &\times \left\{ \left( \frac{a}{n} + k \right) \cdot \left( \frac{a}{n} + \frac{1}{n} + k \right) \cdots \left( \frac{a}{n} + \frac{r}{n} + k \right) \right\}. \end{aligned}$$

The last factor appears because  $n$  does not divide  $j$ . Therefore we have

$$(5.8) \quad (a)_N = n^N \prod_{i=0}^{n-1} \left( \frac{a+i}{n} \right)_k \times \prod_{i=0}^r \left( \frac{a+i}{n} + k \right),$$

where the second product is *not* the Pochhammer symbol. Now employ the relation

$$(5.9) \quad k + c = c \frac{(c+1)_k}{(c)_k},$$

to write

$$(a)_N = n^N \prod_{i=0}^{n-1} \left( \frac{a+i}{n} \right)_k \prod_{i=0}^r \left( \frac{a+i}{n} \right) \cdot \prod_{i=0}^r \left( \frac{a+i}{n} + 1 \right)_k / \prod_{i=0}^r \left( \frac{a+i}{n} \right)_k.$$

This expression reduces to the stated formula.  $\square$

The transformation above yields

$$U_n \left( x^j \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k \right) = x^{1+\lfloor j/n \rfloor} \sum_{k=0}^{\infty} n^{nk(p-q-1)} \frac{(c_1)_k \cdots (c_{pn})_k}{(d_1)_k \cdots (d_{(q+1)n})_k} x^k.$$

The special case  $p = q + 1$  provides a simpler situation, in which the coefficient  $n^{nk(p-q-1)}$  does not appear in the resulting series.

**Theorem 5.6.** Let  $j, n \in \mathbb{N}$  and assume  $p = q + 1$ .

If  $n$  divides  $j$ , we have

$$(5.10) \quad U_n \left( x^j {}_pF_q(\mathbf{a}, \mathbf{b}; x) \right) = x^{j/n} {}_{np}F_{np-1}(\mathbf{c}, \mathbf{d}; x),$$

where  $\mathbf{c}, \mathbf{d}$  are defined in (5.3).

If  $n$  does not divide  $j$ , we have

$$(5.11) \quad U_n(x_p^j F_q(\mathbf{a}, \mathbf{b}; x)) = x^{1+\lfloor j/n \rfloor} {}_{np}F_{np-1}(\mathbf{c}, \mathbf{d}; x),$$

where  $\mathbf{c}, \mathbf{d}$  are defined in (5.4).

## 6. THE EIGENVALUE EQUATION

In this section we focus on the spectral properties of the operator  $U_n$ , as the spectral properties of the operators  $V_n$  are trivial. It is here that we encounter more subtle ideas. We describe first the eigenfunctions of the operator  $U_n$  of the form  $x_p^j F_q(\mathbf{a}, \mathbf{b}; x)$ . That is, we look for parameters  $p, q \in \mathbb{N}$  and complex numbers

$$(6.1) \quad a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q$$

such that, with  $\mathbf{a} = (a_1, \dots, a_p)$  and  $\mathbf{b} = (b_1, \dots, b_q)$ , we have

$$(6.2) \quad U_n(x_p^j F_q(\mathbf{a}, \mathbf{b}; x)) = \lambda x_p^j F_q(\mathbf{a}, \mathbf{b}; x).$$

The results from Theorem 5.1 showed that the action of  $U_n$  on  $x_p^j F_q$  depends on whether or not  $n$  divides  $j$ , which by Theorem 4.2 reduces to the cases  $j = 0$  and  $j = 1$  when considering eigenfunctions of  $U_n$ .

**Case 1:**  $j = 0$ . Under this condition we show that the eigenfunction reduces to a rational function.

**Lemma 6.1.** Assume  $n$  divides  $j$  and that (6.2) has a nontrivial solution. Then, for all  $k \in \mathbb{N}$ , we have

$$(6.3) \quad \prod_{j=1}^p \prod_{i=0}^{n-1} (a_j + nk + i) \times \prod_{j=1}^{q+1} (b_j + k) = \prod_{j=1}^p (a_j + j) \times \prod_{j=1}^{q+1} \prod_{i=0}^{n-1} (b_j + nk + i).$$

*Proof.* Comparing terms of the equation  $U_n f = f$  yields

$$(6.4) \quad \frac{(a_1)_{nk} (a_2)_{nk} \cdots (a_p)_{nk}}{(b_1)_{nk} (b_2)_{nk} \cdots (b_{q+1})_{nk}} = \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_{q+1})_k}.$$

Replace  $k$  by  $k+1$ , divide the two equations and use

$$(6.5) \quad \frac{(a)_{k+1}}{(a)_k} = a + k, \text{ and } \frac{(a)_{n(k+1)}}{(a)_k} = (a + nk)(a + nk + 1) \cdots (a + nk + n - 1)$$

to produce the result.  $\square$

**Lemma 6.2.** Assume  $n$  divides  $j$  and that (6.2) has a nontrivial solution. Then  $p = q + 1$ .

*Proof.* Comparing the degrees of the left and right hand side of (6.4) gives  $pn + q + 1 = p + n(q + 1)$ .  $\square$

**Proposition 6.3.** Assume  $n$  divides  $j$  and that (6.2) has a nontrivial solution. Then

$$\left\{ a_i, \frac{b_i}{n}, \frac{b_i + 1}{n}, \dots, \frac{b_i + n - 1}{n} \right\}_{i=1}^p = \left\{ b_i, \frac{a_i}{n}, \frac{a_i + 1}{n}, \dots, \frac{a_i + n - 1}{n} \right\}_{i=1}^p.$$

*Proof.* The roots of the left and right hand side of (6.4) must match.  $\square$

We now show that the results of this proposition imply that the parameters must match:  $a_i = b_i$  for all indices.

**Proposition 6.4.** Assume  $n$  divides  $j$  and that (6.2) has a nontrivial solution. Then, for any  $k \in \mathbb{N}$ , we have

$$(6.6) \quad \sum_{i=1}^p a_i^k = \sum_{i=1}^p b_i^k.$$

*Proof.* Proof by induction on  $k$ . The case  $k = 1$  comes from matching the coefficients of the next to leading order in  $k$ . Indeed, this matching yields

$$\sum_{i=1}^p a_i + \sum_{j=1}^{n-1} \sum_{i=1}^p (b_i + j) = \sum_{i=1}^p b_i + \sum_{j=1}^{n-1} \sum_{i=1}^p (a_i + j),$$

and the case  $k = 1$  holds. In order to check it for  $k = 2$ , add the squares of the elements in Proposition 6.3 to obtain, from the left hand side the expression

$$\sum_{i=1}^p a_i^2 + \frac{1}{n^2} \sum_{i=1}^p \sum_{j=0}^{n-1} (b_i^2 + 2jb_i + j^2) = \sum_{i=1}^p a_i^2 + \frac{1}{n} \sum_{i=1}^p b_i^2 + \frac{2}{n^2} \sum_{i=1}^p b_i \times \sum_{j=0}^{n-1} j + \frac{p}{n^2} \sum_{j=0}^{n-1} j^2.$$

Matching with the corresponding expression from the right hand side and using the statement for  $k = 1$ , yields

$$(6.7) \quad \sum_{i=1}^p a_i^2 = \sum_{i=1}^p b_i^2.$$

The higher moments can be established along these lines.  $\square$

**Proposition 6.5.** Assume two sets  $\{a_j : 1 \leq j \leq p\}$  and  $\{b_j : 1 \leq j \leq p\}$  of complex numbers satisfy

$$(6.8) \quad \sum_{i=1}^p a_i^k = \sum_{i=1}^p b_i^k,$$

for every  $k \in \mathbb{N}$ . Then, after a possible rearrangement of terms of one of these sets, we have  $a_i = b_i$  for all  $i$ .

*Proof.* For  $\mathbf{a}, \mathbf{b} \in \mathbb{C}^p$  and  $N \in \mathbb{N}$ , let

$$(6.9) \quad \mu_N(\mathbf{a}) = \sum_{i=1}^p a_i^N$$

and let

$$f_{\mathbf{a}}(t) = \sum_{j=0}^{\infty} \mu_j(\mathbf{a}) \frac{t^j}{j!}, \quad f_{\mathbf{b}}(t) = \sum_{j=0}^{\infty} \mu_j(\mathbf{b}) \frac{t^j}{j!}$$

be the generating functions of  $\mu_N(\mathbf{a})$  and  $\mu_N(\mathbf{b})$ , respectively. Now assume  $f_{\mathbf{a}} = f_{\mathbf{b}}$ , i.e.

$$\sum_{j=0}^{\infty} \mu_j(\mathbf{a}) \frac{t^j}{j!} = \sum_{j=0}^{\infty} \mu_j(\mathbf{b}) \frac{t^j}{j!}$$

Expanding further gives

$$\sum_{j=0}^{\infty} \sum_{i=1}^p a_i^j \frac{t^j}{j!} = \sum_{j=0}^{\infty} \sum_{i=1}^p b_i^j \frac{t^j}{j!}$$

Since  $\mu_j$  is defined as a sum of finite terms, we can change the order of summation.

$$\sum_{i=1}^p \sum_{j=0}^{\infty} a_i^j \frac{t^j}{j!} = \sum_{i=1}^p \sum_{j=0}^{\infty} b_i^j \frac{t^j}{j!}$$

This yields

$$(6.10) \quad e^{a_1 t} + e^{a_2 t} + \dots + e^{a_p t} = e^{b_1 t} + e^{b_2 t} + \dots + e^{b_p t}.$$

Suppose first that  $a_i, b_i \in \mathbb{R}$  and order them as

$$a_1 \leq a_2 \leq \dots \leq a_p \text{ and } b_1 \leq b_2 \leq \dots \leq b_p.$$

Eliminate from (6.10) all the terms for which the  $a$ 's and  $b$ 's match, to assume that  $a_1 < b_1$ . Then

$$1 + e^{(a_2 - a_1)t} + \dots + e^{(a_p - a_1)t} = e^{(b_1 - a_1)t} + e^{(b_2 - a_1)t} + \dots + e^{(b_p - a_1)t}.$$

Finally, let  $t \rightarrow -\infty$  to get a contradiction.  $\square$

We summarize the previous discussion in the following Theorem.

**Theorem 6.6.** Suppose there exists an eigenfunction of  $U_n$  of the form

$$(6.11) \quad f(x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_{q+1})_k} x^k$$

corresponding to the eigenvalue  $\lambda$  (this is the case  $j = 0$  in (6.2)). Then  $\lambda = 1$ ,  $p = q + 1$  and  $a_i = b_i$  for all  $i$ . Therefore  $f(x) = \frac{1}{1-x}$ .

Proposition 4.2 shows that the only possible values for  $j$  are 0 or 1. The remainder of the section treats the case  $j = 1$ .

**Case 2:**  $j = 1$ . Under this condition we show that the spectrum of the operator  $U_n$  is the set  $\{n^i : i \in \mathbb{Z}\}$ . The corresponding eigenfunctions are the polylogarithm functions

$$(6.12) \quad \text{PolyLog}_i(x) := \sum_{k=1}^{\infty} k^i x^k,$$

corresponding to the eigenvalue  $n^i$  with negative  $i$ , and the eigenfunctions

$$\left(x \frac{d}{dx}\right)^i \left(\frac{1}{1-x}\right)$$

corresponding to the eigenvalue  $n^i$  with non-negative  $i$ .



**Example 6.7.** The dilogarithm function

$$(6.13) \quad \text{Li}_2(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

satisfies

$$(6.14) \quad U_n(\text{Li}_2(x)) = \frac{1}{n^2} \text{Li}_2(x).$$

Therefore  $1/n^2 \in \text{Spec}(U_n)$ . The dilogarithm function admits the hypergeometric representation

$$(6.15) \quad \text{Li}_2(x) = x {}_3F_2 \left( \begin{pmatrix} 1 & & 1 \\ & 2 & 2 \end{pmatrix}; x \right)$$

We now explore properties of eigenfunctions of the operator  $U_n$ .

**Proposition 6.8.** Assume (6.2) has a nontrivial solution with  $j = 1$ . Then, for all  $k \in \mathbb{N}$ , we have

$$(6.16) \quad \prod_{j=1}^p (a_j + k - 1) \cdot \prod_{j=1}^{q+1} \prod_{i=-1}^{n-2} (b_j + nk + i) = \prod_{j=1}^{q+1} (b_j + k - 1) \cdot \prod_{j=1}^p \prod_{i=-1}^{n-2} (a_j + nk + i).$$

*Proof.* Assume the eigenfunction has the form

$$(6.17) \quad f(x) = x \sum_{k=0}^{\infty} c_k x^k.$$

Comparing coefficients in the equation  $U_n f = \lambda f$  gives

$$(6.18) \quad c_{nk-1} = \lambda c_{k-1}.$$

Replacing the standard hypergeometric type yields

$$(6.19) \quad \frac{(a_1)_{nk-1} \cdots (a_p)_{nk-1}}{(b_1)_{nk-1} \cdots (b_{q+1})_{nk-1}} = \lambda \frac{(a_1)_{k-1} \cdots (a_p)_{k-1}}{(b_1)_{k-1} \cdots (b_{q+1})_{k-1}}.$$

Replace  $k$  by  $k + 1$  and divide the two corresponding equations to obtain the result.  $\square$

**Lemma 6.9.** Assume  $j = 1$  and that (6.2) has a nontrivial solution of the form

$$(6.20) \quad f(x) = x \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k.$$

Then  $p = q + 1$ .

*Proof.* Compare the degrees on both sides of the polynomial in Lemma 6.8.  $\square$

**Proposition 6.10.** Assume

$$(6.21) \quad f(x) = x \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_p)_k} x^k$$

is an eigenfunction for  $U_n$ . Then

$$\bigcup_{i=1}^p \left\{ a_i - 1, \frac{b_i - 1}{n}, \frac{b_i}{n}, \dots, \frac{b_i + n - 2}{n} \right\} = \bigcup_{i=1}^p \left\{ b_i - 1, \frac{a_i - 1}{n}, \frac{a_i}{n}, \dots, \frac{a_i + n - 2}{n} \right\}.$$

*Proof.* These are the roots of both sides of the polynomial in Lemma 6.8.  $\square$

We now show that this equality of sets imposes severe restrictions on the eigenvalues and eigenfunctions of the operator  $U_n$ . We discuss first the eigenvalues.

**Proposition 6.11.** Assume

$$(6.22) \quad f(x) = x \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_p)_k} x^k$$

is an eigenfunction for  $U_n$ . Let

$$(6.23) \quad \gamma_a := |\{i \in \{1, 2, \dots, p\} : a_i = 1\}|,$$

and

$$(6.24) \quad \gamma_b := |\{i \in \{1, 2, \dots, p\} : b_i = 1\}|.$$

Then

$$(6.25) \quad n^{\gamma_a} (a_1)_{n-1} (a_2)_{n-1} \cdots (a_p)_{n-1} = n^{\gamma_b} (b_1)_{n-1} (b_2)_{n-1} \cdots (b_p)_{n-1}.$$

*Proof.* Consider the product of all the non-zero terms on the left-hand side of Proposition 6.10. The removal of the zero terms, corresponding to those  $a_i = 1$  carry with them the removal of a power of  $n$ .  $\square$

**Example 6.12.** In example 6.7 we have  $\gamma_a = 3$  and  $\gamma_b = 1$ . The statement (6.25) states that

$$(6.26) \quad n^3 \times (1)_{n-1} (1)_{n-1} (1)_{n-1} = n \times (2)_{n-1} (2)_{n-1} (1)_{n-1},$$

which is correct.

**Theorem 6.13.** Assume  $j = 1$  and

$$(6.27) \quad f(x) = x \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_p)_k}$$

is an eigenfunction for  $U_n$  with eigenvalue  $\lambda$ . Then

$$(6.28) \quad \lambda = n^{\gamma_b - \gamma_a},$$

with  $\gamma_a, \gamma_b$  defined in (6.25).

*Proof.* Put  $k = 1$  in the relation (6.19) to obtain

$$(6.29) \quad \lambda = \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_{q+1})_{n-1}}.$$

Now use (6.25) to conclude.  $\square$

The result above shows that the spectrum of  $U_n$  satisfies

$$(6.30) \quad \text{Spec}(U_n) \subset \{n^a : a \in \mathbb{Z}\}.$$

The example below shows that we actually have equality.

**Example 6.14.** Let  $i \in \mathbb{Z}$ . Then the hypergeometric series

$$(6.31) \quad f_i(x) := \sum_{k=1}^{\infty} k^i x^k,$$

is an eigenfunction of  $U_n$ , with eigenvalue  $n^i$ . The dilogarithm corresponds to the case  $i = -2$ .

**Theorem 6.15.** The spectrum of  $U_n$  is the set  $\{n^i : i \in \mathbb{Z}\}$ .

We now discuss the eigenfunctions of  $U_n$ , starting with the sets

$$\bigcup_{i=1}^p \left\{ a_i - 1, \frac{b_i - 1}{n}, \frac{b_i}{n}, \dots, \frac{b_i + n - 2}{n} \right\} = \bigcup_{i=1}^p \left\{ b_i - 1, \frac{a_i - 1}{n}, \frac{a_i}{n}, \dots, \frac{a_i + n - 2}{n} \right\},$$

that have appeared in Proposition 6.10. Recall that  $\{a_i, b_i\}$  are the parameters of the eigenfunction

$$(6.32) \quad f(x) = x \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k.$$

We may assume that  $a_i \neq b_j$ , otherwise the term  $(a_i)_k = (b_j)_k$  should be cancelled. Observe that if we let  $c_i = a_i - 1$  and  $d_i = b_i - 1$ , the basic set identity in Case 2 becomes the basic set identity of Case 1, with  $c_i$  instead of  $a_i$  and  $d_i$  instead of  $b_i$ . The reason why one cannot deduce  $a_i = b_i$  is that some of the  $a_i$  in Case 2 could be 1, and the corresponding  $c_i$  would vanish. This violates the basic assumption of Proposition 6.5.

We bypass this difficulty by defining

$$(6.33) \quad a'_i = \begin{cases} a_i & \text{if } a_i \neq 1, \\ 2 & \text{if } a_i = 1, \end{cases}$$

and

$$(6.34) \quad b'_i = \begin{cases} b_i & \text{if } b_i \neq 1, \\ 2 & \text{if } b_i = 1. \end{cases}$$

The sets described above, obtained by replacing  $a_i, b_i$  by  $a'_i, b'_i$  remain equal. In order to see this, observe that if  $a_1 = 1$ , then the elements containing  $a_1$  are

$$(6.35) \quad \left\{ \frac{a_1 - 1}{n}, \frac{a_1}{n}, \dots, \frac{a_1 + n - 2}{n} \right\},$$

are replaced by

$$(6.36) \quad \left\{ \frac{a_1}{n}, \frac{a_1 + 1}{n}, \dots, \frac{a_1 + n - 1}{n} \right\},$$

which is equivalent to simply replacing  $a_1$  by  $a_1 + 1$ . Theorem 6.6 shows that  $a'_i = b'_i$ . If both  $a_i$  and  $b_i$  are equal to 1 or both different from 1, we conclude that  $a_i = b_i$ . This contradicts our original assumption. In the case  $a'_i = b'_i$ , with  $a_i = 1$ , we conclude that  $b_i = 2$ . Similarly, if  $a'_i = b'_i$ , with  $b_i = 1$ , we obtain  $a_i = 2$ . Therefore, each  $a_i = 1$  produces the valid pair  $\{1, 2\}$ , and each  $b_i = 1$  leads to  $\{2, 1\}$ . Using the fact that there are no common parameters from top and bottom, we conclude that an eigenfunction must have the following structure.

**Theorem 6.16.** Assume

$$(6.37) \quad f(x) = x \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_{q+1})_k} x^k$$

is an eigenfunction of  $U_n$ , with the usual normalization  $b_{q+1} = 1$ . Then, either  $a_i = 2$  and  $b_i = 1$  for all  $1 \leq i \leq p$ , or  $a_i = 1$  and  $b_i = 2$  for all  $1 \leq i \leq p$ .

In other words, if  $f$  is an eigenfunction of any particular Hecke operator  $U_n$ , then

$$(6.38) \quad f(x) = {}_aF_{a-1}(1_a, 2_{a-1}; x) = \sum_{k=0}^{\infty} \frac{1}{k^a} x^k,$$

or

$$(6.39) \quad f(x) = {}_aF_{a-1}(2_a, 1_{a-1}; x) = \sum_{k=0}^{\infty} k^a x^k,$$

where  $a \in \mathbb{N}$ ,  $1_a$  stands for the  $a$ -vector  $(1, 1, \dots, 1)$  and  $2_b$  is defined analogously.

## 7. THE SIMULTANEOUS EIGENFUNCTIONS OF $U_n$ FOR ALL $n$

In this section we completely characterize those hypergeometric functions which are simultaneous eigenfunctions of  $U_n$  for all  $n$ , and give an application to the theory of completely multiplicative functions.

What do the hypergeometric functions  $f(x) = \sum_{k=1}^{\infty} c_k x^k$  that are simultaneous eigenfunctions of all of the linear operators  $U_n$  look like? It turns out there is a simple answer: they are precisely the polylogarithms and the rational functions  $(x \frac{d}{dx})^a \left( \frac{1}{1-x} \right)$ , as given by the following theorem.

**Theorem 7.1.** Let

$$(7.1) \quad f(x) = \sum_{k=1}^{\infty} c_k x^k$$

be a hypergeometric function with no constant term. Then  $f$  is a simultaneous eigenfunction for the set of all Hecke operators  $\{U_n\}_{n=1}^{\infty}$  with respective eigenvalues  $\{\lambda_n\}_{n=1}^{\infty}$  if and only if

$$(7.2) \quad f(x) = C \sum_{k=1}^{\infty} k^a x^k,$$

with  $a \in \mathbb{Z}$  and  $C \in \mathbb{C}$ . In other words,  $f$  is a polylogarithm, or  $f = (x \frac{d}{dx})^a \left( \frac{1}{1-x} \right)$ .

□

Before proving this theorem, we prove an interesting corollary regarding a number theoretic fact concerning hypergeometric coefficients that are completely multiplicative functions of the summation index.

**Corollary 7.2.** The hypergeometric coefficient

$$(7.3) \quad c(n) = \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}},$$

is a completely multiplicative function of  $n$  if and only if it is of the form  $Cn^a$  for some  $a \in \mathbb{Z}$  and  $C \in \mathbb{C}$ .

*Proof.* By the definition of the completely multiplicative function  $c(n)$ , we know that

$$(7.4) \quad c(nk) = c(n)c(k)$$

for all  $k, n \in \mathbb{N}$ . This implies that

$$(7.5) \quad f(x) = \sum_{k=1}^{\infty} c(k)x^k$$

is an eigenfunction of  $U_n$ , with eigenvalue  $c(n)$ . This holds for all  $n$ , thus it is a simultaneous eigenfunction. The result now follows from Theorem 7.1.  $\square$

We recall that by definition  $f$  is a simultaneous eigenfunction of all of the Hecke operators if and only if for all  $n \in \mathbb{N}$ ,

$$(7.6) \quad U_n f = \lambda_n f.$$

Treating  $f$  as a power series without even considering its hypergeometric properties, we have

$$(7.7) \quad \sum_{k=1}^{\infty} c_{nk} x^k = \lambda_n \sum_{k=1}^{\infty} c_k x^k$$

so that

$$(7.8) \quad c_{nk} = \lambda_n c_k$$

This is true for all  $k \geq 1$ , so we can set  $k = 1$  which gives us

$$(7.9) \quad c_n = \lambda_n c_1$$

This, in turn, is true for all  $n$ , which proves the following.

**Lemma 7.3.** Let

$$(7.10) \quad f(x) = \sum_{k=1}^{\infty} c_k x^k$$

be a power series with no constant term that is a simultaneous eigenfunction for the set of operators  $\{U_n\}_{n=1}^{\infty}$  with respective eigenvalues  $\{\lambda_n\}_{n=1}^{\infty}$ . Then, for  $i \geq 2$ , we have

$$(7.11) \quad c_i = \lambda_i c_1$$

and thus

$$(7.12) \quad f(x) = c_1 \sum_{k=1}^{\infty} \lambda_k x^k.$$

$\square$

*Proof.* (of Theorem 7.1) Let us suppose now that  $f$ , as defined in (7.12), is a hypergeometric function. By theorem 6.13, we have that

$$(7.13) \quad \lambda_n = n^{\gamma_b - \gamma_a}$$

where  $\gamma_b$  and  $\gamma_a$  are defined in 6.25. This proves theorem 7.1.  $\square$

In our formal hypergeometric notation, we see that any simultaneous eigenfunction must be the polylog

$$(7.14) \quad f(x) = \sum_{k=0}^{\infty} \frac{1}{k^a} x^k,$$

or the rational function

$$(7.15) \quad g(x) = \sum_{k=0}^{\infty} k^a x^k,$$

where  $a$  is any nonnegative integer. In conclusion, we see that if a hypergeometric function is an eigenfunction for a single operator  $U_j$ , then it is automatically a simultaneous eigenfunction for all of the Hecke operators  $U_n$ , as  $n$  varies over all positive integers. This situation lies in sharp contrast with the space of rational functions studied recently in [5], [9], and [11].

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DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118  
*E-mail address:* `vhm@math.tulane.edu`

DIVISION OF MATHEMATICAL SCIENCES, NANYANG TECHNOLOGICAL UNIVERSITY, SPMS-MAS-03-01, 21 NANYANG LINK, SINGAPORE 637371  
*E-mail address:* `rsinai@ntu.edu.sg`

DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PA 19122  
*E-mail address:* `ksoodha@temple.edu`